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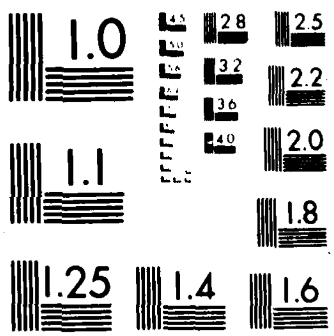
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BY

M. E. BOCK, P. DIACONIS, F. HUFFER

TECHNICAL REPORT NO. 379
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1. INTRODUCTION

There is a naturally conjectured inequality for sums of random variables that seems hard to prove. Consider Y_i, $1 \leq i \leq n$, independent and identically distributed positive random variables. Let

$$W = \theta_1 Y_1 + \theta_2 Y_2 + \cdots + \theta_n Y_n$$

with θ_i positive constants. The conjecture is this: among all ways of varying θ_i that preserve the mean (so $\theta_1 + \cdots + \theta_n$ stays fixed), the tails of W are smallest when all θ_i are equal.

It is not hard to show that the variance of W (or the expected value of any convex function) is smallest when all θ_i are equal. It often seems as if much more is true.

For symmetric distributions a fairly general result is known. Let f be the common density of $Y_1, \ldots, Y_n$. Define $\underset{\sim}{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$. Proschan (1965) showed that if f is symmetric (about zero) and $\log f$ is concave, then $P[W \geq t]$ is Schur-convex in $\underset{\sim}{\theta}$ for $t > 0$.

For positive random variables (having asymmetric distributions) very little is known. We have chosen to work mainly with the gamma distribution. Our results are summarized in the following.

Let Y_i have a Gamma(α,β) density $f(y) = e^{-\beta y} y^{\alpha-1} \beta^{\alpha} / \Gamma(\alpha)$ on $(0,\infty)$. For $n=2$ we show

$$P[W \leq t] \text{ is Schur-convex in } \underset{\sim}{\theta} \text{ if}$$

$$t \leq \alpha(\theta_1+\theta_2)/\beta, \quad \text{and}$$

$P[W \geq t]$ is Schur-convex in $\underset{\sim}{\theta}$ if

$$t \geq (\alpha + \tfrac{1}{2})(\theta_1 + \theta_2)/\beta .$$

Specializing to a standard exponential with $\alpha=\beta=1$, both tails are smallest when $\theta_1=\theta_2$. Here the left tail starts any place left of the mean $(= \theta_1+\theta_2)$, and the right tail starts any place to the right of the mean times 3/2. No simple convexity holds between these two values.

For general $n > 2$, our results go in the same direction although they are cruder. For a sum of n Gamma(α,β) variables

$P[W \leq t]$ is Schur-convex for $\underset{\sim}{\theta}$ in the region $\{\underset{\sim}{\theta}: \min_{1 \leq i \leq n} \theta_i \geq t\beta/(n\alpha+1)\}$, and

$P[W \geq t]$ is Schur-convex in $\underset{\sim}{\theta}$ for

$$t \geq (n\alpha+1)(\theta_1+\theta_2+\cdots+\theta_n)/\beta .$$

Our results cover positive linear combinations of chi-square variables. These arise in the asymptotic distributions of nonparametric goodness of fit tests. See Chernoff and Lehmann (1954) and Moore (1978) and Alvo, Cabilio and Fiegen (1982) (for the average Kendall tau statistic). The results of this paper apply because a linear combination of the form $c_1\chi_n^2 + c_2\chi_m^2$ may be written as

$$\sum_{i=1}^{n+m} \theta_i Y_i$$

where $\theta_i = 2c_1$ for $i=1,\ldots,n$ and $\theta_i = 2c_2$ for $i = n+1,\ldots,n+m$, and the independent gamma variables Y_i each have shape parameter $\alpha = \frac{1}{2}$ and $\beta = 1$.

Positive linear combinations of chi-square variables also arise when positive linear combinations of sample variances from normal populations are formed.

Linear combinations of chi-square variables can also occur as limiting distributions of U-statistics. For example, let $h(x,y)$ be a symmetric function with associated U-statistic

$$U_n = \frac{1}{\binom{n}{2}} \sum_{i<j} h(X_i,X_j)$$

where the X_i are independent identically distributed random variables with distribution function F. Assume $E(h^2(X_1,X_2))$ is positive and finite. Define $h_1(x)$ to be $E(h(x,X_1))$ and ξ_1 to be the variance of $h_1(X_2)$. Let $\theta = Eh(X_1,X_2)$. If ξ_1 is zero, then $n(U_n-\theta)$ converges in distritubion to a constant plus a linear combination of independent chi-square random variables with one degree of freedom. Details and examples can be found in Gregory (1977) or Serfling (1980) Section 5.5.2.

Our results may be applied to study linear combinations of exponential variables. These arise, for example, as waiting time distributions in pure birth processes (see Feller (1971), page 41).

We also give results for the Weibull distribution when $n = 2$: Let the Y_i's have a Weibull (β,γ) density

$$f(y) = \gamma\,\beta y^{\beta-1} e^{-\gamma y^{\beta}} I_{(0,\infty)}(y) .$$

We show that $P[W \leq t_0]$ is Schur-concave in $\underset{\sim}{\theta}$ if

$$\text{(a)} \quad t_0 \geq (\theta_1+\theta_2)((1+\frac{1}{2\beta})/\gamma)^{\frac{1}{\beta}} \qquad \text{when} \quad \beta \geq 1$$

or

$$\text{(b)} \quad t_0 \geq (\theta_1+\theta_2)((1+\frac{1}{2\beta})/\gamma)^{\frac{1}{\beta}} \, 2^{\frac{1-\beta}{\beta}} \qquad \text{when} \quad 0 < \beta < 1 .$$

Our results show that the tail of the linear combination W is minimized when all the coefficients θ_i are equal. This is analogous to results for binomial random variables in the master's thesis of Chebychev and in the work of Hoeffding (1956) and Gleser (1975) which are described in Section K of Chapter 12 of Marshall and Olkin (1979). (Our results resemble Gleser's but the inequalities go in the opposite direction.) We believe similar results hold much more generally, e.g., for all naturally occurring exponential families. We have been unable to provide a general theory.

It is possible to consider the question of Schur-convexity or concavity in other metrics for the θ_i's and it should be noted that Tong (1980) has shown for gamma random variables that $P(W \leq t)$ is strictly Schur-concave in $(\theta_1^{-1},\ldots,\theta_n^{-1})$ for all positive t and $n \geq 2$.

In Section 2 of this paper we treat the case of a positive linear combination of two independent identically distributed gamma random variables. In Section 3 we look at this case for a Weibull distribution.

In Section 4 we examine the distribution function of a positive linear combination of n independent identically distributed gamma random variables.

2. POSITIVE LINEAR COMBINATIONS OF TWO GAMMA VARIABLES

In this section we set $n=2$. Without loss of generality we assume $\theta_1+\theta_2 = 1$ and assume the scale factor β is one in the Gamma (α,β) distribution. Then with $p = \theta_1$ we have

$$W = pY_1 + (1-p)Y_2$$

where $0 < p < 1$ and the Y_i are independent random variables each with density

$$f(x) = x^{\alpha-1} e^{-x}/\Gamma(\alpha) ,$$

for $x > 0$, where α is positive.

In the case $n = 2$ the Schur-convexity or Schur-concavity in $\underset{\sim}{\theta}$ of the distribution function of W is determined by examining the sign of the following derivative

$$\frac{\partial}{\partial p} P(pY_1 + (1-p)Y_2 \leq t) ,$$

where $\frac{1}{2} < p < 1$.

We show for all p there exists a value $t_0(p)$ such that the derivative is positive for t less than $t_0(p)$ and negative for t

greater than $t_0(p)$. Also $t_0(p)$ lies between α and $\alpha + \frac{1}{2}$. Thus $P(W \leq t)$ is Schur-concave in θ for $t \leq \alpha + \frac{1}{2}$ and Schur-convex for $t \leq \alpha$. Furthermore $P(W \leq t)$ is neither Schur-concave nor Schur-convex in θ for $\alpha < t < \alpha + \frac{1}{2}$. This result was observed empirically by Solomon (1961) in distribution function tables for the case of $\alpha = \frac{1}{2}$ (when the gamma variable is a multiple of a chi-square variable with one degree of freedom). The result was shown analytically by Diaconis in unpublished results for $\alpha = 1$ (when the gamma variable is an exponential random variable). See Marshall and Olkin (1979), Chapter 12, Theorem K.3.

Notation: For the rest of this section Y_1, Y_2 are independent gamma random variables with density $x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ and p will always satisfy $\frac{1}{2} < p < 1$.

Furthermore, we will often write

$$F_p(t) = P[pY_1 + (1-p)Y_2 \leq t] .$$

Lemma 1: The sign of

$$\frac{\partial}{\partial p} P[pY_1 + (1-p)Y_2 \leq t]$$

is the same as the sign of $S_p(t)$ defined by

$$S_p(t) = \int_0^1 (p-x)x^{\alpha-1}(1-x)^{\alpha-1}e^{\lambda tx}dx$$

where $\lambda = \frac{2p-1}{p(1-p)}$. In fact,

$$\frac{\partial}{\partial p} P[pY_1 + (1-p)Y_2 \le t] = KS_p(t)$$

where $K = \left(\frac{t^2}{p(1-p)}\right)^{\alpha+1} \frac{e^{-t/(1-p)}}{(t\Gamma(\alpha))^2}$.

Proof:

$$F_p(t) = \int_0^{\frac{t}{p}} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} \int_0^{\frac{t-py}{(1-p)}} \frac{u^{\alpha-1}e^{-u}}{\Gamma(\alpha)} \, du dy$$

implies

$$\frac{\partial}{\partial p} F_p(t) = \int_0^{\frac{t}{p}} \frac{(t-y)}{(1-p)^2} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} \left(\frac{t-py}{1-p}\right)^{\alpha-1} \frac{e^{-\left(\frac{t-py}{1-p}\right)}}{\Gamma(\alpha)} \, dy \; .$$

Making the change of variable $x = \frac{p}{t} y$ yields

$$\frac{\partial}{\partial p} F_p(t) = \frac{t}{p} \int_0^1 \frac{(t-tx/p)}{(1-p)^2} \left(\frac{tx}{p}\right)^{\alpha-1} \frac{e^{-\frac{tx}{p}}}{\Gamma(\alpha)} \left(\frac{t-tx}{1-p}\right)^{\alpha-1} \frac{e^{-\left(\frac{t-tx}{1-p}\right)}}{\Gamma(\alpha)} \, dx$$

$$= \left(\frac{t^2}{p(1-p)}\right)^{\alpha+1} \frac{e^{-\frac{t}{(1-p)}}}{(t\Gamma(\alpha))^2} \int_0^1 (p-x)[x(1-x)]^{\alpha-1} e^{\lambda tx} \, dx$$

where

$$\lambda = -\frac{1}{p} + \frac{1}{(1-p)} = \frac{2p-1}{p(1-p)} . \qquad \text{QED}$$

Remark: The next lemma shows that for $\frac{1}{2} < p < 1$, the function $e^{-\lambda pt}S_p(t)$ is strictly decreasing in t and thus can have at most one change of sign. From the definition of $S_p(t)$ it is clear that $S_p(0) > 0$ and that $S_p(t) < 0$ for all sufficiently large t. Thus there exists $t_0(p) > 0$ such that

$$\frac{\partial}{\partial p} F_p(t) > 0 \quad \text{for} \quad t < t_0(p) \quad \text{and}$$

$$\frac{\partial}{\partial p} F_p(t) < 0 \quad \text{for} \quad t > t_0(p) .$$

Lemma 2: For $\frac{1}{2} < p < 1$, the function $e^{-\lambda pt}S_p(t)$ is strictly decreasing in t.

Proof:

$$e^{-\lambda pt}S_p(t) = \int_0^1 (p-x)x^{\alpha-1}(1-x)^{\alpha-1}e^{\lambda t(x-p)}dx$$

so that

$$\frac{\partial}{\partial t}[e^{-\lambda pt}S_p(t)] = -\lambda \int_0^1 (p-x)^2x^{\alpha-1}(1-x)^{\alpha-1}e^{\lambda t(x-p)}dx$$

which is negative. QED

The next lemmas 3, 4, and 5 are used in the proof of Theorem 1 which states that $t_0(p)$ is less than $(\alpha + \frac{1}{2})$.

Lemma 3: $\frac{\partial}{\partial p} F_p(t)$ has the same sign as

$$\int_0^1 \frac{e^{-t/h(s)}}{(h(s))^{2\alpha+1}} [(1-2s)s^{\alpha-1}(1-s)^{\alpha-1}]ds$$

where $h(s) = ps + (1-p)(1-s)$.

Proof:

$$F_p(t) = \iint \frac{x^{\alpha-1}e^{-x}}{\Gamma(\alpha)} \frac{y^{\alpha-1}e^{-y}}{\Gamma(\alpha)} dxdy ,$$

where integration is over the values such that $x \geq 0$, $y \geq 0$ and $px + (1-p)y \leq t$.

Substituting $r = x+y$ and $s = x/(x+y)$ this becomes

$$F_p(t) = \Gamma(\alpha)^{-2} \int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}ds \int_0^{t/h(s)} r^{2\alpha-1}e^{-r}dr$$

so that

$$\frac{\partial}{\partial p} F_p(t) = \frac{t^{2\alpha}}{\Gamma(\alpha)^2} \int_0^1 \frac{e^{-t/h(s)}}{(h(s))^{2\alpha+1}} [(1-2s)s^{\alpha-1}(1-s)^{\alpha-1}]ds . \qquad \text{QED}$$

Lemma 4: Define

$$f(x,y) = \frac{\log(\frac{1}{x}) - \log(\frac{1}{y})}{(\frac{1}{x} - \frac{1}{y})}$$

for $x \neq y$ and $f(x,x) = x$. Then $f(x,y)$ is strictly Schur-concave.

<u>Proof</u>: Note that

$$\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right) f(x,y) = \left(\frac{x+y}{y-x}\right) - \left(\frac{x^2+y^2}{(y-x)^2}\right) \log(y/x) .$$

So for $x < y$ we must show that

$$\frac{x+y}{y-x} < \left(\frac{x^2+y^2}{(y-x)^2}\right) \log(y/x)$$

or equivalently

$$\frac{y^2-x^2}{x^2+y^2} < \log(y/x)$$

or equivalently (substituting $u = y/x$)

$$\frac{u^2-1}{u^2+1} < \log u \text{ for } u > 1 .$$

Both sides are equal when $u = 1$ so it suffices to differentiate both sides and show that

$$\frac{4u}{(u^2+1)^2} < \frac{1}{u} \quad \text{for} \quad u > 1 .$$

This is immediate. QED

<u>Lemma 5</u>: For $x > 0$ define $g(x) = \frac{1}{x} + \log x$.

(a) If $0 < y < z$ and $y+z \leq 2$, then $g(y) > g(z)$.

(b) If $g(y+z) \geq g(y-z)$, then $g(y+u) > g(y-u)$ for $0 < u < z$.

Proof:

(a) For $y < z$, $g(y) > g(z)$ is equivalent to $f(y,z) < 1$ where f is the function from the previous lemma. By Schur-concavity

$$f(y,z) < f(\frac{y+z}{2}, \frac{y+z}{2}) = (y+z)/2$$

so that $g(y) > g(z)$ when $(y+z)/2 \leq 1$.

(b) For $r < s$, $g(r) < g(s)$ if and only if $f(r,s) > 1$. Similarly $g(r) \leq g(s)$ if and only if $f(r,s) \geq 1$. Thus $g(y+z) \geq g(y-z)$ implies $f(y-z, y+z) \geq 1$. Now Schur-concavity yields $f(y-u,y+u) > f(y-z,y+z) \geq 1$ for $0 < u < z$. Thus $g(y+u) > g(y-u)$ as desired. QED

Theorem 1: (See Remark preceding lemma 2).

For $p > \frac{1}{2}$,

$$t_0(p) < \alpha + \frac{1}{2} \text{ ;}$$

$$\text{that is, } \frac{\partial}{\partial p} F_p(t) < 0 \quad \text{for} \quad t \geq \alpha + \frac{1}{2} \text{ .}$$

Proof: We use the result of lemma 3. The function $(1-2s)s^{\alpha-1}(1-s)^{\alpha-1}$ is odd about the value $s = \frac{1}{2}$. Thus, to show that $\frac{\partial}{\partial p} F_p(t) < 0$, it suffices to show that

$$\frac{e^{-t/h(s)}}{(h(s))^{2\alpha+1}} < \frac{e^{-t/h(1-s)}}{(h(1-s))^{2\alpha+1}}$$

for $0 < s < \frac{1}{2}$. By taking logs and doing some algebra, this is seen to be equivalent to

$$\frac{c}{h(s)} + \log\left(\frac{h(s)}{c}\right) > \frac{c}{h(1-s)} + \log\left(\frac{h(1-s)}{c}\right)$$

where $c = \frac{t}{2\alpha+1}$. Now use lemma 5(a). Since the function $h(s) = ps + (1-p)(1-s)$ is increasing for $p > \frac{1}{2}$, the previous inequality will hold if

$$\frac{h(s)}{c} + \frac{h(1-s)}{c} \leq 2 .$$

But $h(s) + h(1-s) = 1$ for all s, so this is equivalent to $c \geq \frac{1}{2}$ or $t \geq \alpha + \frac{1}{2}$. QED

The following theorem gives a useful bound on $t_0(p)$ for small p or for small α.

<u>Theorem 2</u>: For $p > \frac{1}{2}$ we have $t_0(p) > (\alpha + \frac{1}{2})\,\psi(p)$ where

$$\psi(p) = 2\left(\frac{\log(\frac{1}{1-p}) - \log(\frac{1}{p})}{\frac{1}{1-p} - \frac{1}{p}}\right) .$$

We have $\Psi(p) < 1$ for $p > \frac{1}{2}$ and $\lim_{p\to\frac{1}{2}} \Psi(p) = 1$. A small table is given below.

p	$\psi(p)$
.55	.9933
.60	.9731
.65	.9389
.70	.8897
.75	.8240
.80	.7394
.85	.6319
.90	.4944

For $\alpha = \frac{1}{2}$, $p = .80$, the tables of Solomon (1960) give $t_0(p) \approx .79$ and the bound is .7394.

Proof: We repeat the argument of the earlier theorem (proving $t_0(p) < \alpha + \frac{1}{2}$) but with the inequalities reversed.

To show $\frac{\partial}{\partial p} F_p(t) > 0$, it suffices to show

$$(*) \qquad \frac{c}{h(s)} + \log\left(\frac{h(s)}{c}\right) < \frac{c}{h(1-s)} + \log\left(\frac{h(1-s)}{c}\right)$$

for $0 < s < \frac{1}{2}$ where $c = \frac{t}{2\alpha+1}$. Observe that the function $h(s)$ is increasing and $h(s) + h(1-s) = 1$. Thus lemma 5 part (b) says that $(*)$ will be true if

$$\frac{c}{h(0)} + \log\left(\frac{h(0)}{c}\right) \leq \frac{c}{h(1)} + \log\left(\frac{h(1)}{c}\right) .$$

Using $h(0) = 1-p$, $h(1) = p$ and a little algebra this becomes

$$\frac{\log(\frac{1}{1-p}) - \log(\frac{1}{p})}{(\frac{1}{1-p} - \frac{1}{p})} \geq c . \qquad \text{QED}$$

The following lemma will be needed in the proof of Theorem 3.

Lemma 6:

$$\int_0^1 [\lambda tx(1-x) + \alpha(1-2x)]x^{\alpha-1}(1-x)^{\alpha-1}\, e^{\lambda tx}dx = 0\ .$$

Proof:

$$\frac{\partial}{\partial x}\,[x^{\alpha}(1-x)^{\alpha}] = \alpha(1-2x)x^{\alpha-1}(1-x)^{\alpha-1}$$

so that integration by parts gives

$$\int_0^1 [x^{\alpha}(1-x)^{\alpha}][\lambda te^{\lambda tx}]dx = -\int_0^1 [\alpha(1-2x)x^{\alpha-1}(1-x)^{\alpha-1}]e^{\lambda tx}dx\ .$$

Moving both integrals to the same side of the equation and then combining them into one integral completes the proof.   QED

Theorem 3: For $t < \alpha$ and $\frac{1}{2} < p < 1$,

$$\frac{\partial}{\partial p}\, F_p(t) > 0\ .$$

Proof: We have earlier used the fact (Lemma 1) that $\frac{\partial}{\partial p} F_p(t)$ has the same sign as

$$S_p(t) = \int_0^1 (p-x)x^{\alpha-1}(1-x)^{\alpha-1}e^{\lambda tx}dx$$

where $\lambda = \frac{2p-1}{p(1-p)}$. Therefore it suffices to show that $S_p(t) > 0$

for $t < \alpha$. This can be achieved by noting that $S_p(t)$ has the same sign as $e^{\phi}S_p(t)$ where ϕ is any smooth function depending on p and t. Now $S_p(0)$ is positive and if $e^{\phi}S_p(t)$ is increasing in t for $t < \alpha$, then $S_p(t)$ must be positive for $t < \alpha$. So it suffices to show that for some smooth function ϕ we have $\frac{\partial}{\partial t}(e^{\phi}S_p(t)) > 0$ for $t < \alpha$.

For any function ϕ we have

$$\frac{\partial}{\partial t}(e^{\phi}S_p(t)) = \int_0^1 (p-x)(\lambda x+k)x^{\alpha-1}(1-x)^{\alpha-1}e^{\lambda tx}dx$$

with $k = \frac{\partial\phi}{\partial t}$. By the preceding lemma we may replace $(p-x)(\lambda x+k)$ in the above integral by

$$(*) \qquad (p-x)(\lambda x+k) + c[\lambda tx(1-x) + \alpha(1-2x)]$$

without changing the value of the integral. Here c is any quantity whose value does not depend on x but may depend on t and p. By appropriate choice of c and k, the expression (*) can be made into a constant, that is, an expression not involving x. Choosing $c = -1/t$ eliminates the quadratic (x^2) term. Next we choose $k = (2\alpha/t) - \lambda(1-p)$ to eliminate the linear term. With these choices the constant term becomes $pk + c\alpha = (2p-1)(\frac{\alpha}{t} - 1) > 0$ for $t < \alpha$. Thus $\frac{\partial}{\partial t}(e^{\phi}S_p(t)) > 0$ for $t < \alpha$ where ϕ is determined by $\frac{\partial\phi}{\partial t} = (2\alpha/t) - \lambda(1-p)$. QED

3. POSITIVE LINEAR COMBINATIONS OF TWO WEIBULL VARIABLES

In this section it is shown that a positive linear combination $(\theta_1 Y_1 + \theta_2 Y_2)$ of two independent identically distributed Weibull random variables Y_i has a distribution function which is Schur-concave in $\underset{\sim}{\theta}$ for sufficiently large argument. Without loss of generality we set $\theta_1 + \theta_2 = 1$ and we set the scale factor of the Weibull distribution to one. Then $P(\theta_1 Y_1 + \theta_2 Y_2 \leq t)$ is Schur-concave in $\underset{\sim}{\theta}$ if

(a) $\qquad t \geq (1 + \frac{1}{2\beta})^{\frac{1}{\beta}} \qquad$ when $\beta \geq 1$

or

(b) $\qquad t \geq (1 + \frac{1}{2\beta})^{\frac{1}{\beta}} \; 2^{\frac{1-\beta}{\beta}} \qquad$ when $0 < \beta \leq 1$.

Again the Schur-concavity will follow if the derivative with respect to p of the distribution function $P[pY_1 + (1-p)Y_2 \leq t]$ is negative in an appropriate region. Theorem 4 developes conditions under which the derivative is negative and Corollary 1 interprets them to provide the result above.

Notation: Define

$$F_p(t) = P[pY_1 + (1-p)Y_2 \leq t] .$$

Theorem 4: Let Y_1, Y_2 be independent Weibull random variables with density given by $\beta x^{\beta-1} \exp(-x^{\beta})$. If $\frac{1}{2} < p < 1$ and

$$t^{\beta} \geq (1 + \frac{1}{2\beta})M ,$$

then $\frac{\partial}{\partial p} F_p(t) < 0$. Here

$$M = \begin{cases} 1 & \text{for } \beta \geq 1 \\ p^{\beta} + (1-p)^{\beta} & \text{for } \beta \leq 1 . \end{cases}$$

Corollary 1: Under the assumptions of Theorem 4, we have $\frac{\partial}{\partial p} F_p(t) < 0$ *if*

(a) $\quad t \geq (1 + \frac{1}{2\beta})^{\frac{1}{\beta}} \quad$ when $\beta \geq 1$

(b) $\quad t \geq (1 + \frac{1}{2\beta})^{\frac{1}{\beta}}\, 2^{\frac{1-\beta}{\beta}} \quad$ when $0 < \beta \leq 1$.

Proof: Part (a) follows immediately from the theorem. It is necessary to find an upper bound for $p^{\beta} + (1-p)^{\beta}$ independent of p for part (b). Because for $\frac{1}{2} < p < 1$,

$$\{p^{\beta} + (1-p)^{\beta}\}$$

is a decreasing function of p, the maximum occurs for $p = \frac{1}{2}$, i.e.

$$\{(\tfrac{1}{2})^{\beta} + (\tfrac{1}{2})^{\beta}\} = (\tfrac{1}{2})^{\beta-1} = 2^{1-\beta} .$$

Thus $t^{\beta} \geq (1 + \frac{1}{2\beta})2^{1-\beta}$, i.e. $t \geq (1 + \frac{1}{2\beta})^{\frac{1}{\beta}}\, 2^{\frac{1-\beta}{\beta}}$. QED

The following lemma is used in the proof of Theorem 4.

Lemma 7: For independent Weibull variables Y_i with density given in Theorem 4, we have

$$\frac{\partial}{\partial p} F_p(t) = \beta^2 t^{2\beta} \int_0^1 (1-2s)s^{\beta-1}(1-s)^{\beta-1} \left[\frac{\exp\left(\frac{-t^\beta \phi(s)}{h(s)}\right)}{h(s)^{2\beta+1}}\right] ds$$

where $\phi(s) = s^\beta + (1-s)^\beta$ and $h(s) = ps + (1-p)(1-s)$.

Proof of Lemma: Change from Cartesian coordinates to r,s, defined by $r = x+y$, $s = \frac{x}{x+y}$. Then

$$F_p(t) = \int_0^1 \int_0^{t/h(s)} [rf(rs)f(r(1-s))] dr ds$$

and

$$\frac{\partial}{\partial p} F_p(t) = \int_0^1 (1-2s) \frac{t^2}{h(s)^3} [f(\frac{st}{h(s)}) \ f(\frac{(1-s)t}{h(s)})] ds$$

where $h(s) = ps + (1-p)(1-s)$ and f is the Weibull density. QED

Proof of Theorem 4: Use Lemma 7 and let g denote the expression in brackets inside the integral for $\frac{\partial}{\partial p} F_p(t)$. The function $(1-2s)s^{\beta-1}(1-s)^{\beta-1}$ is odd about the value $s = \frac{1}{2}$. Therefore, to prove $\frac{\partial}{\partial p} F_p(t) < 0$, it suffices to show that $g(s) < g(1-s)$ for $0 < s < \frac{1}{2}$. Taking logs this becomes

$$\frac{t^\beta \phi(s)}{h(s)^\beta} + (2\beta+1)\log h(s) > \frac{t^\beta \phi(1-s)}{h(1-s)^\beta} + (2\beta+1)\log h(1-s)$$

or equivalently

$$\frac{c\phi(s)}{h(s)^\beta} + \log\left(\frac{h(s)^\beta}{c\phi(s)}\right) > \frac{c\phi(1-s)}{h(1-s)^\beta} + \log\left(\frac{h(1-s)^\beta}{c\phi(1-s)}\right)$$

where $c = \frac{\beta t^\beta}{2\beta+1}$. To obtain this last expression we have used the fact that $\phi(s) = \phi(1-s)$. By Lemma 5(a) (concerning the function $\frac{1}{x} + \log x$) we need only show

(i) $\quad \frac{h(s)^\beta}{c\phi(s)} < \frac{h(1-s)^\beta}{c\phi(1-s)}$, and

(ii) $\quad \frac{h(s)^\beta}{c\phi(s)} + \frac{h(1-s)^\beta}{c\phi(1-s)} \le 2$

for $0 < s < \frac{1}{2}$. Note that (i) is true because h is strictly increasing when $p > \frac{1}{2}$ and (ii) will be true if

$$\underset{0 \le s \le \frac{1}{2}}{\text{Maximum}} \left(\frac{h(s)^\beta + h(1-s)^\beta}{s^\beta + (1-s)^\beta}\right) \le 2c\ .$$

In the lemma which follows, this maximum is shown to equal 1 for $\beta \ge 1$ and to equal $p^\beta + (1-p)^\beta$ for $\beta \le 1$. Now use $c = \frac{\beta t^\beta}{2\beta+1}$ and the proof is complete. QED

<u>Lemma 8</u>: Define

$$\psi(s) = \frac{h(s)^\beta + h(1-s)^\beta}{s^\beta + (1-s)^\beta}\ .$$

Then

$$\underset{0 \le s \le \frac{1}{2}}{\text{Maximum}}\ \psi(s) = \begin{cases} 1 & \text{for } \beta \ge 1 , \\ p^\beta + (1-p)^\beta & \text{for } \beta \le 1 . \end{cases}$$

<u>Proof</u>: Remember that $h(s) = ps + (1-p)(1-s)$ and $\frac{1}{2} < p < 1$.

Consider first $\beta < 1$. A stationary point must satisfy $\frac{\partial}{\partial s} \log \psi(s) = 0$ which is equivalent to

$$(*) \qquad \frac{\beta(2p-1)(h(s)^{\beta-1} - h(1-s)^{\beta-1})}{h(s)^\beta + h(1-s)^\beta} = \frac{\beta(s^{\beta-1} - (1-s)^{\beta-1})}{s^\beta + (1-s)^\beta} .$$

But for $0 < s < \frac{1}{2}$ it is easy to verify that $h(s)^\beta + h(1-s)^\beta > s^\beta + (1-s)^\beta$ and $h(s)^{\beta-1} - h(1-s)^{\beta-1} < s^{\beta-1} - (1-s)^{\beta-1}$. Since $0 < 2p-1 < 1$, the left hand side of (*) is strictly less than the right hand side. Thus ψ has no stationary point in the open interval $(0,\frac{1}{2})$ and ψ must achieve its maximum at 0 or $\frac{1}{2}$.

$$\psi(0) = p^\beta + (1-p)^\beta > 1 = \psi(\tfrac{1}{2}) .$$

Now consider $\beta \ge 1$.

$$\frac{\partial}{\partial p} [h(s)^\beta + h(1-s)^\beta] = \beta(1-2s)[h(1-s)^{\beta-1} - h(s)^{\beta-1}] \ge 0$$

for $0 \le s \le \frac{1}{2}$. Therefore $\frac{\partial}{\partial p} \psi_p(s) \ge 0$ for $0 \le s \le \frac{1}{2}$. The subscript on ψ indicates the value of p. Since $\psi_p(s)$ increases with p we have $\psi_p(s) \le \psi_1(s) = 1 = \psi_p(\frac{1}{2})$ as desired. QED

4. POSITIVE LINEAR COMBINATION OF N GAMMA VARIABLES

We show that the distribution function of a positive linear combination of n independent gamma random variables is Schur-concave in the coefficients when the argument of the distribution function is sufficiently large. This implies that the right tail probabilities are Schur-convex. The "sufficiently large" bound for the argument appears to be too large when compared with the precise results of Section 2 for $n = 2$ and can probably be improved. Corollary 2 shows that for independent Y_i each with density $x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ and positive θ_i, $i = 1,\ldots,n$, $P(\sum_{i=1}^{n} \theta_i Y_i \le t)$ is Schur-concave in θ for $t \ge (n\alpha + 1) \sum_{i=1}^{n} \theta_i$. For $n = 2$ and $\theta_1 = p$ and $\theta_2 = 1-p$ where $0 < p < 1$ this lower bound for t is $(2\alpha+1)$ which is too large by a factor of 2 according to the results of Section 2.

<u>Theorem 5</u>: Let $Y_1, Y_2, \ldots, Y_n$ be i.i.d. gamma random variables with density $x^{\alpha-1}e^{-x}/\Gamma(\alpha)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ where $\theta_i > 0$ for all i. Define

$$P_t(\theta) = P[\sum_{i=1}^{n} \theta_i Y_i \le t] .$$

As a function of θ, $P_t(\theta)$ is symmetric and convex (and therefore Schur-convex) inside the region $\{\theta : t \le (n\alpha+1)\mathrm{Min}\ \theta_i\}$; $P_t(\theta)$ is symmetric and concave (and therefore Schur-concave) inside the region $\{\theta : t \ge (n\alpha+1)\mathrm{Max}\ \theta_i$.

Corollary 2: $P_t(\theta)$ is Schur-concave in θ for $t \geq (n\alpha+1) \sum_{i=1}^{n} \theta_i$.

Proof: Because $\max \theta_i \leq \sum_{i=1}^{n} \theta_i$, we have that $t \geq (n\alpha+1) \sum_{i=1}^{n} \theta_i$ implies that $t \geq (n\alpha+1)\max \theta_i$. Theorem 5 implies that $P_t(\theta)$ is symmetric and concave for θ inside the region $\{\theta : t \geq (n\alpha+1) \sum_{i=1}^{n} \theta_i\}$. Thus $P_t(\theta)$ is Schur-concave there. QED

In what follows, t is held fixed at some arbitrary positive value. The basic tool in the proof of Theorem 5 is the following result due essentially to Marhshall and Proschan (1965). (See Chapter 11, page 288 of Marshall and Olkin (1979).)

Proposition: If $X_1, X_2, \ldots, X_n$ are exchangeable and g is a continuous convex function, then $\psi(a_1, a_2, \ldots, a_n) = Eg(\sum_{i=1}^{n} a_i X_i)$ is symmetric and convex. It is also true upon replacing convex by concave.

Proof of Theorem 5: For all i define $S_i = Y_i / (\sum_{k=1}^{n} Y_k)$. The vector $(S_1, S_2, \ldots, S_n)$ is exchangeable and independent of $\sum_{i=1}^{n} Y_i$. These are standard properties.

Define $F(u) = P[\sum_{i=1}^{n} Y_i \leq h]$

$$= \int_0^u x^{n\alpha-1} e^{-x} dx / \Gamma(n\alpha).$$

Then

$$P[\sum_{i=1}^{n} \theta_i Y_i \le t] = P[(\sum_{i=1}^{n} \theta_i S_i)(\sum_{i=1}^{n} Y_i) \le t]$$

$$= EP[(\sum\theta_i S_i)(\sum Y_i) \le t | \sum\theta_i S_i]$$

$$= EF(t/\sum\theta_i S_i) = Eg(\sum\theta_i S_i)$$

where $g(u) = F(t/u)$. A short calculation shows that

$$g''(u) = [(n\alpha+1) - \frac{t}{u}] \frac{t^{n\alpha} e^{-t/u}}{\Gamma(\alpha) u^{n\alpha+2}} .$$

Thus g is convex for $u \ge \frac{t}{n\alpha+1}$ and concave for $0 < u \le \frac{t}{n\alpha+1}$. For θ in the region $\{\theta : t \le (n\alpha+1) \mathrm{Min}\, \theta_i\}$,

$$\sum \theta_i S_i \ge \frac{t}{n\alpha+1} .$$

Thus $Eg(\sum\theta_i S_i)$ is symmetric and convex by the proposition. Similarly, for θ in $\{\theta : t \ge (n\alpha+1) \mathrm{Max}\, \theta_i\}$,

$$\sum \theta_i S_i \le \frac{t}{n\alpha+1}$$

and therefore $Eg(\sum\theta_i S_i)$ is symmetric and concave. QED

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We ~~study~~ the behavior of the tail probabilities of weighted averages of certain i.i.d. random variables as the weights are varied. We show that the upper and lower tails are smallest when all the weights are equal. Our results apply to exponential, chi-square, gamma and Weibull random variables.

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